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The elliptical Hertzian contact of transversely isotropic magnetoelectroelastic bodies

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Abstract

First, the general solution for transversely isotropic magnetoelectroelastic media is given concisely in form of five harmonic functions. Second, the extended Boussinesq and Cerruti solutions for the magnetoelectroelastic half-space are obtained in terms of elementary functions by utilizing this general solution. Third, the coupled fields for elliptical Hertzian contact of magnetoelectroelastic bodies are solved in smooth and frictional cases. At last, the graphic results are presented.

Keywords: Hertzian contact; Transversely isotropic; Magnetoelectroelastic

1. Introduction

Because the stress concentration near the contact region, which is caused by inharmonious contact between the components, could cause the component failure. Therefore, it is necessary to make theoretical analysis and accurate quantitative description of the behavior of solids under contact.

For purely elastic isotropic and transversely isotropic solids, since Hertz (1882) published his classic article "On the contact of elastic solids", the researches on the contact of elastic materials have been conducted for more than 100 years, and a lot of scientists contributed to this area (Elliott, 1948, 1949; Mindlin, 1949; Sneddon, 1951; Shield, 1951; Muskhelishvili, 1953; Green and Zerna, 1954; Willis, 1966, 1967; Conway et al., 1967; Conway and Farnham, 1967; Chen, 1969; Pan and Chou, 1976; Keer and Mowry, 1979; Gladwell, 1980; Johnson, 1985; Fabrikant, 1989, 1991; Lin et al., 1991; Hanson, 1992a,b, 1994; Hanson and Puja, 1997).

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For transversely isotropic piezoelectric materials with electromechanical coupling, Fan et al. (1996) studied the two-dimensional contact on a piezoelectric half-plane by using Stroh's formalism, and gave the solutions for loads acting on the boundary of an anisotropic piezoelectric half-plane. Ding et al. (1999) gave the electroelastic field of elliptical Hertzian contact of transversely isotropic piezoelectric bodies. Ding et al. (2000), Chen and Ding (1999), Chen et al. (1999), Chen (1999, 2000), Giannakopoulos (2000) and Sridhar et al. (2000) studied a series of contact problems such as spherical contact, conical indentation and upright or tilted circular flat punch on a transversely isotropic piezoelectric half-space, and obtained their electroelastic fields.

To the author's knowledge, no work has been done regarding the study of contact problem for media possessing simultaneously piezoelectric, piezomagnetic and magnetoelectric effects, namely, magnetoelectroelastic solids. A wide class of crystals (Alshits et al., 1992) and the emerging composite materials that are made from the piezoelectric media and piezomagnetic media (Huang and Kuo, 1997; Li and Dunn, 1998) do have these mixed properties.

In the present paper, the general solution for transversely isotropic magnetoelectroelastic media is given concisely in form of five harmonic functions. And then, the extended Boussinesq and Cerruti solutions for the magnetoelectroelastic half-space are obtained in terms of elementary functions by utilizing this general solution. Third, aiming at elliptical Hertzian contact of magnetoelectroelastic bodies, we solve for its coupled fields in smooth and frictional cases by first evaluating the displacement functions and then differentiating. The displacement functions can be obtained by integrating the extended Boussinesq or Cerruti solutions in the contact region. At last, when only normal pressure is loaded, the elastic and electric fields in the magnetoelectroelastic half-space are compared in the figures with those of corresponding piezoelectric and purely elastic half-spaces. In addition, the magnetic field in the magnetoelectroelastic half-space are also shown in the figures.

2. General solution for transversely isotropic magnetoelectroelastic media

As suggested by Huang and Kuo (1997), the governing equations for the theory of magnetoelectroelasticity are:

$$\sigma_{ij,j} = -f_i, \tag{1}$$

$$D_{j,j} = \rho_{\mathrm{f}}, \tag{2}$$

$$B_{j,j} = 0, \tag{3}$$

$$\sigma_{ij} = C_{ijkl}\bar{\varepsilon}_{kl} - e_{kij}E_k - d_{kij}H_k, \tag{4}$$

$$D_i = e_{ikl}\bar{\varepsilon}_{kl} + \varepsilon_{ik}E_k + g_{ik}H_k, \tag{5}$$

$$B_i = d_{ikl}\bar{\varepsilon}_{kl} + g_{ik}E_k + \mu_{ik}H_k, \tag{6}$$

$$\bar{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{7}$$

$$E_i = -\Phi_{,i}, \tag{8}$$

$$H_i = -\Psi_{,i}, \tag{9}$$

where σ_{ij}, $\bar{\varepsilon}_{ij}$, u_i, E_i, D_i, H_i and B_i are the components of stress, strain, displacement, electric field, electric displacement, magnetic field and magnetic induction, respectively; Φ and Ψ are the electric potential and magnetic potential, respectively; f_i and ρ_f are body force, and density of free charges, respectively; c_{ijkl}, e_{kij}, d_{kij}, ε_{ij}, g_{ij} and μ_{ij} are elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic constants, respectively.

For the transversely isotropic magnetoelectroelastic media whose isotropic plane are perpendicular to the z-axis of Cartesian coordinates (x, y, z), the dependent physical constants are elastic constants c_{11}, c_{12}, c_{13}, c_{33}, c_{44}; piezoelectric constants e_{31}, e_{33}, e_{15}; piezomagnetic constants d_{31}, d_{33}, d_{15}; dielectric constants ε_{11}, ε_{33}; electromagnetic constants g_{11}, g_{33} and magnetic constants μ_{11}, μ_{33}. In addition, elastic constant $c_{66} = (c_{11} - c_{12})/2$. In the absence of body forces and free charges, substituting Eqs. (4)–(6) into Eqs. (1)–(3), we obtain five equilibrium equations which are expressed in terms of u, v, w, Φ and Ψ. And then, based on these equilibrium equations, Eqs. (7)–(9) and the method which Ding et al. (1996) used to solve the coupled equations of piezoelectric media, we can obtain the general solution of displacement, electric potential and magnetic potential in terms of five displacement functions ψ_j $(j = 0, 1, 2, 3, 4)$, which satisfy, respectively, the following equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_j^2}\right)\psi_j = 0 \quad (j = 0, 1, 2, 3, 4), \tag{10}$$

where $z_j = s_j z$ $(j = 0, 1, 2, 3, 4)$, $s_0 = \sqrt{c_{66}/c_{44}}$ and s_j $(j = 1, 2, 3, 4)$ are the four characteristic roots of an eighth degree equation defined as follows and satisfy $\mathrm{Re}(s_j) > 0$

$$a_1 s^8 - a_2 s^6 + a_3 s^4 - a_4 s^2 + a_5 = 0, \tag{11}$$

where a_k $(k = 1, 2, 3, 4, 5)$ are listed in Appendix A.

Using the constitutive equations (4)–(6), the general solutions for the stress, electric displacement and magnetic induction expressed by five displacement functions are obtained. At this point, the coefficients in front of derivatives of displacement functions with respect to coordinates are all products or linear combinations of material constants and characteristic roots. If expressions for the stress, electric displacement and magnetic induction are substituted into five equilibrium equations, some relations among these coefficients will be determined through consideration of Eq. (10). With these relations taken into account, the general solutions for stress, electric displacement and magnetic induction can be obtained.

For the sake of convenience, the following notations are introduced:

$$\begin{aligned}
&U = u + \mathrm{i}v = \mathrm{e}^{\mathrm{i}\phi}(u_r + \mathrm{i}u_\phi),\\
&w_1 = w, \quad w_2 = \Phi, \quad w_3 = \Psi,\\
&\sigma_1 = \sigma_x + \sigma_y = \sigma_r + \sigma_\phi,\\
&\sigma_2 = \sigma_x - \sigma_y + 2\mathrm{i}\tau_{xy} = \mathrm{e}^{2\mathrm{i}\phi}(\sigma_r - \sigma_\phi + 2\mathrm{i}\tau_{r\phi}),\\
&\tau_{z1} = \tau_{xz} + \mathrm{i}\tau_{yz} = \mathrm{e}^{\mathrm{i}\phi}(\tau_{rz} + \mathrm{i}\tau_{\phi z}), \quad \sigma_{z1} = \sigma_z,\\
&\tau_{z2} = D_x + \mathrm{i}D_y = \mathrm{e}^{\mathrm{i}\phi}(D_r + \mathrm{i}D_\phi), \quad \sigma_{z2} = D_z,\\
&\tau_{z3} = B_x + \mathrm{i}B_y = \mathrm{e}^{\mathrm{i}\phi}(B_r + \mathrm{i}B_\phi), \quad \sigma_{z3} = B_z.
\end{aligned} \tag{12}$$

By virtue of Eq. (12), all components in Cartesian coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z) can be transformed to each other easily. Then, the general solution can be concisely written as follows:

$$
\begin{aligned}
&U = \Lambda\left(\mathrm{i}\psi_0 + \sum_{j=1}^{4}\psi_j\right), \quad w_m = \sum_{j=1}^{4} s_j k_{mj}\frac{\partial \psi_j}{\partial z_j},\\
&\sigma_1 = 2\sum_{j=1}^{4}(c_{66} - \omega_{1j}s_j^2)\frac{\partial^2\psi_j}{\partial z_j^2} = -2\sum_{j=1}^{4}(c_{66} - \omega_{1j}s_j^2)\Lambda\psi_j,\\
&\sigma_2 = 2c_{66}\Lambda^2\left(\mathrm{i}\psi_0 + \sum_{j=1}^{4}\psi_j\right),\\
&\sigma_{zm} = \sum_{j=1}^{4}\omega_{mj}\frac{\partial^2\psi_j}{\partial z_j^2} = -\sum_{j=1}^{4}\omega_{mj}\Lambda\psi_j,\\
&\tau_{zm} = \Lambda\left(s_0\rho_m\mathrm{i}\frac{\partial\psi_0}{\partial z_0} + \sum_{j=1}^{4}s_j\omega_{mj}\frac{\partial\psi_j}{\partial z_j}\right) \quad (m = 1,2,3),
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
&\omega_{1j} = s_j^2(c_{33}k_{1j} + e_{33}k_{2j} + d_{33}k_{3j}) - c_{13}, \quad \rho_1 = c_{44},\\
&\omega_{2j} = s_j^2(e_{33}k_{1j} - \varepsilon_{33}k_{2j} - g_{33}k_{3j}) - e_{31}, \quad \rho_2 = e_{15},\\
&\omega_{3j} = s_j^2(d_{33}k_{1j} - g_{33}k_{2j} - \mu_{33}k_{3j}) - d_{31}, \quad \rho_3 = d_{15},\\
&k_{mj} = \frac{\beta_{mj}}{\alpha_j s_j^2}, \quad \Lambda = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \quad (m = 1,2,3;\ j = 1,2,3,4),
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
&\alpha_j = -n_1 + n_2 s_j^2 - n_3 s_j^4,\\
&\beta_{mj} = -n_{4m} + n_{5m}s_j^2 - n_{6m}s_j^4 + n_{7m}s_j^6 \quad (m = 1,2,3),
\end{aligned} \tag{15}
$$

where n_1, n_2, n_3, m_{4m}, m_{5m}, m_{6m} and m_{7m} are listed in Appendix B.

Thus, the general solutions for transversely isotropic magnetoelectroelastic media are obtained.

3. The solutions for point forces and point charge acting on the half-space of magnetoelectroelastic media

Considering a transversely isotropic magnetoelectroelastic half-space $z \geqslant 0$ where the surface $z = 0$ is parallel to the planes of isotropy, the extended Boussinesq and Cerruti solutions for point forces and point charge acting on the surface of this half-space are derived in this section.

3.1. The extended Boussinesq solution for normal point force P_z and point charge Q acting on the coordinate origin

This is an axisymmetric problem. Functions ψ_0 and ψ_j can be assumed in the following form:

$$
\psi_0 = 0, \quad \psi_j = A_j \ln R_j^* \quad (j = 1,2,3,4), \tag{16}
$$

where $R_j^* = R_j + z_j$, $R_j = \sqrt{r^2 + z_j^2}$ and $r^2 = x^2 + y^2$, A_j are undetermined constants.

Substituting Eq. (16) into Eq. (13), we have

$$\begin{aligned}
&U=\sum_{j=1}^{4}A_j\frac{x+\mathrm{i}y}{R_jR_j^*},\quad w_m=\sum_{j=1}^{4}A_js_jk_{mj}\frac{1}{R_j},\\
&\sigma_1=2c_{66}\sum_{j=1}^{4}A_j\left[\frac{2}{R_jR_j^*}-\frac{x^2+y^2}{R_j^2R_j^*}\left(\frac{1}{R_j}+\frac{1}{R_j^*}\right)\right]-2\sum_{j=1}^{4}A_j(2c_{66}-\omega_{1j}s_j^2)\frac{z_j}{R_j^3},\\
&\sigma_2=2c_{66}(y^2-x^2)\sum_{j=1}^{4}A_j\frac{1}{R_j^2R_j^*}\left(\frac{1}{R_j}+\frac{1}{R_j^*}\right)-4\mathrm{i}c_{66}xy\sum_{j=1}^{4}A_j\frac{1}{R_j^2R_j^*}\left(\frac{1}{R_j}+\frac{1}{R_j^*}\right),\\
&\sigma_{zm}=-\sum_{j=1}^{4}A_j\omega_{mj}\frac{z_j}{R_j^3},\quad \tau_{zm}=-\sum_{j=1}^{4}A_js_j\omega_{mj}\frac{x+\mathrm{i}y}{R_j^3}.
\end{aligned}\tag{17}$$

The boundary conditions at $z=0$ require

$$\tau_{z1}=0,\quad \sigma_{zm}=0 \qquad (m=1,2,3).\tag{18}$$

Obviously, $\sigma_{zm}=0$ are satisfied automatically. Substituting Eq. (17) into $\tau_{z1}=0$, we have

$$\sum_{j=1}^{4}s_j\omega_{1j}A_j=0.\tag{19}$$

Meanwhile, taking into consideration all the equilibrium conditions, apart from those already satisfied, for the layer cut from the infinite magnetoelectroelastic half-space by the two planes $z=0$ and $z=h$. we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sigma_{zm}(x,y,h)\,\mathrm{d}x\,\mathrm{d}y+P_m=0\quad (m=1,2,3),\tag{20}$$

where

$$P_1=P_z,\quad P_2=-Q,\quad P_3=0,\tag{21}$$

Substituting σ_{zm} in Eq. (17) into Eq. (20), we have

$$2\pi\sum_{j=1}^{4}\omega_{mj}A_j=P_m\quad (m=1,2,3),\tag{22}$$

Combining Eqs. (19) and (22) to determine A_j, we obtain

$$A_j=\delta_jP_z+\lambda_jQ,\tag{23}$$

where

$$\begin{aligned}
&\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{Bmatrix}=\begin{bmatrix}s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} & s_4\omega_{14}\\ 2\pi\omega_{11} & 2\pi\omega_{12} & 2\pi\omega_{13} & 2\pi\omega_{14}\\ 2\pi\omega_{21} & 2\pi\omega_{22} & 2\pi\omega_{23} & 2\pi\omega_{24}\\ 2\pi\omega_{31} & 2\pi\omega_{32} & 2\pi\omega_{33} & 2\pi\omega_{34}\end{bmatrix}^{-1}\begin{Bmatrix}0\\ 1\\ 0\\ 0\end{Bmatrix},\\
&\begin{Bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{Bmatrix}=\begin{bmatrix}s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} & s_4\omega_{14}\\ 2\pi\omega_{11} & 2\pi\omega_{12} & 2\pi\omega_{13} & 2\pi\omega_{14}\\ 2\pi\omega_{21} & 2\pi\omega_{22} & 2\pi\omega_{23} & 2\pi\omega_{24}\\ 2\pi\omega_{31} & 2\pi\omega_{32} & 2\pi\omega_{33} & 2\pi\omega_{34}\end{bmatrix}^{-1}\begin{Bmatrix}0\\ 0\\ -1\\ 0\end{Bmatrix}.
\end{aligned}\tag{24}$$

Substituting Eq. (23) into Eqs. (19) and (22), because P_0 and Q_0 can be arbitrary value, we can get following identities:

$$\begin{aligned}
&\sum_{j=1}^{4} s_j\omega_{1j}\delta_j = 0, \quad \sum_{j=1}^{4} s_j\omega_{1j}\lambda_j = 0,\\
&\sum_{j=1}^{4} \omega_{1j}\delta_j = \frac{1}{2\pi}, \quad \sum_{j=1}^{4} \omega_{1j}\lambda_j = 0,\\
&\sum_{j=1}^{4} \omega_{2j}\delta_j = 0, \quad \sum_{j=1}^{4} \omega_{2j}\lambda_j = -\frac{1}{2\pi},\\
&\sum_{j=1}^{4} \omega_{3j}\delta_j = 0, \quad \sum_{j=1}^{4} \omega_{3j}\lambda_j = 0.
\end{aligned} \tag{25}$$

In order to study the contact problems of piezoelectric materials, the displacement w of a point on the surface, which is at a distance of r from the origin, is given as follow.

$$w = \sum_{j=1}^{4} \frac{A_j s_j k_{1j}}{R_j} = \frac{KP_z + LQ}{r}, \tag{26}$$

where

$$K = \sum_{j=1}^{4} s_j k_{1j}\delta_j, \quad L = \sum_{j=1}^{4} s_j k_{1j}\lambda_j, \tag{27}$$

Eq. (26) shows that the displacement w on the surface is in inverse proportion to r.

3.2. The extended Cerruti solution for tangential point forces P_x and P_y acting on the coordinate origin

Functions ψ_0 and ψ_j can be assumed in the following form:

$$\psi_0 = \frac{B_0 y}{R_0^*} - \frac{C_0 x}{R_0^*}, \quad \psi_j = \frac{B_j x}{R_j^*} + \frac{C_j y}{R_j^*} \quad (j = 1, 2, 3, 4), \tag{28}$$

where B_0 and B_j are undetermined constants.

After some work parallel to Section 3.1, B_j and C_j can be determined as follow:

$$B_j = P_x\eta_j, \quad C_j = P_y\eta_j \qquad (j = 0, 1, 2, 3, 4), \tag{29}$$

where

$$\eta_0 = -\frac{1}{2\pi s_0 c_{44}}, \quad \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{Bmatrix} = \frac{1}{2\pi}\begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} & s_4\omega_{14} \\ \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{30}$$

3.3. The displacement functions for point forces and point charge acting on arbitrary point on the surface

When cylindrical coordinates (r, ϕ, z) is adopted, and the point charge Q and three point forces P_x, P_y and P_z with their positive directions same as x-, y- and z-axes act on arbitrary point $M(r_0, \phi_0, 0)$ on the surface of a transversely isotropic magnetoelectroelastic half-space, the displacement functions are listed as follows.

3.3.1. Point charge Q and normal point force P_z are loaded

According to Eqs. (16) and (23), we have

$$\begin{aligned} &\psi_0(r, \phi, z; r_0, \phi_0) = 0, \\ &\psi_j(r, \phi, z; r_0, \phi_0) = (P_z\delta_j + Q\lambda_j)\ln R_j^* \quad (j = 1, 2, 3, 4), \end{aligned} \tag{31}$$

where

$$R_j^* = R_j + z_j, \quad R_j = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0) + z_j^2} \tag{32}$$

and constants δ_j and λ_j are expressed in Eq. (24).

3.3.2. Tangential forces P_x and P_y are loaded

According to Eqs. (28) and (29), we have

$$\begin{aligned} &\psi_0(r, \phi, z; r_0, \phi_0) = \mathrm{i}G_0(P\bar{\Delta} - \overline{P}\Delta)\chi(z_0), \\ &\psi_j(r, \phi, z; r_0, \phi_0) = G_j(P\bar{\Delta} + \overline{P}\Delta)\chi(z_j) \quad (j = 1, 2, 3, 4), \end{aligned} \tag{33}$$

where

$$\chi(z_j) = z_j \ln R_j^* - R_j \quad (j = 0, 1, 2, 3, 4), \tag{34}$$

$$G_j = -\eta_j/2 \quad (j = 0, 1, 2, 3, 4), \tag{35}$$

where $P = P_x + \mathrm{i}P_y$ is complex shear force, $\overline{P}$ and $\bar{\Delta}$ are the complex conjugate of P and Δ, respectively. η_j, R_j and R_j^* are expressed in Eqs. (30) and (32).

4. The contact region and contact loads for contact between a magnetoelectroelastic body and another body under forces and charges

As shown in Fig. 1, body ① (which is magnetoelectroelastic solid or others such as piezoelectric and purely elastic solid) and a magnetoelectroelastic body ② are pressed to each other by a pair of forces P_z. Meanwhile, a pair of charges $+Q$ and $-Q$ locate at two points on the common normal line and in body ① and body ②, respectively.

Analysis can be taken same as Ding et al. (2000). Assume that

(1) The shape of contact region S is elliptical and its dimensions are sufficiently small compared with those of the bodies ① and ②, so we can regard them as two half-spaces. S is defined as follows:

$$S : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{36}$$

(2) The normal electric displacement on the surface of bodies ① and ② is nonzero only inside the contact region S. The contact pressure $p(x, y)$ and electric displacement $d(x, y)$ inside the contact region distribute in following form.

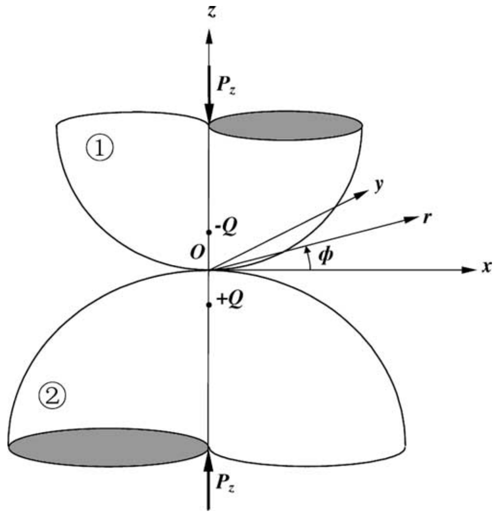

Fig. 1. Contact of body ① and body ②.

$$p(x,y) = p_0\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}, \qquad \text{within } S: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{37}$$
$$d(x,y) = d_0\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2},$$

where

$$a = n_a\left(\frac{c_pP_z+c_dQ}{\Sigma}\right)^{1/3}, \quad b = n_b\left(\frac{c_pP_z+c_dQ}{\Sigma}\right)^{1/3},$$
$$p_0 = n_pP_z\left(\frac{\Sigma}{c_pP_z+c_dQ}\right)^{2/3}, \quad d_0 = n_pQ\left(\frac{\Sigma}{c_pP_z+c_dQ}\right)^{2/3} \tag{38}$$

and

$$n_a = \left[\frac{3}{\pi}\left(1+\frac{B}{A}\right)D(e)\right]^{1/3}, \quad n_b = \left\{\frac{3}{\pi}\left(1+\frac{B}{A}\right)[K(e)-D(e)]\sqrt{1-e^2}\right\}^{1/3}, \quad n_p = \frac{3}{2\pi n_a n_b}, \tag{39}$$
$$c_p = (K_1+K_2)\pi, \quad c_d = (L_2-L_1)\pi, \quad \Sigma = K_{11}+K_{12}+K_{21}+K_{22}.$$

It is noted that

(1) For magnetoelectroelastic bodies, K_n and L_n can be obtained from Eq. (27) as follows:

$$K_n = \left(\sum_{j=1}^{4} s_jk_{1j}\delta_j\right)_n, \quad L_n = \left(\sum_{j=1}^{4} s_jk_{1j}\lambda_j\right)_n, \tag{40}$$

where subscripts $n = 1, 2$ correspond to bodies ① and ②; when body ① is transversely isotropic piezoelectric medium (the magnetic field is uncoupled from electroelastic field), according to Ding et al. (2000), we have

$$K_1 = \sum_{j=1}^{3} s_j k_{1j} \delta_j, \quad L_1 = \sum_{j=1}^{3} s_j k_{1j} \lambda_j, \tag{41}$$

where s_j and k_{1j} ($j = 1, 2, 3$) are defined in Eqs. (32) and (41) of Ding et al. (1996). δ_j and λ_j are defined in Eq. (10) of Ding et al. (2000); when body ① is purely elastic transversely isotropic medium (the elastic field is uncoupled from electroelastic field), $L_1 = 0$, and according to Ding et al. (1997), we have

$$K_1 = \frac{(s_1 + s_2)c_{11}}{2\pi s_1 s_2 (c_{11}c_{33} - c_{13}^2)}, \tag{42}$$

where c_{ij} are elastic constants, s_k ($k = 1, 2$) are defined in Hu (1953); when body ① is purely elastic rigid body, $L_1 = 0$, $K_1 = 0$.

(2) K_{11}, K_{12} and K_{21}, K_{22} are the principal curvatures of bodies ① and ② at the original point.

(3) $e = \sqrt{1 - (b/a)^2}$ is the eccentricity of the ellipse S, and

$$K(e) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}, \quad E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \varphi \, \mathrm{d}\varphi},$$
$$D(e) = \frac{K(e) - E(e)}{e^2}, \quad \frac{B}{A} = \frac{K(e) - D(e)}{(1 - e^2)D(e)}. \tag{43}$$

5. The magnetoelectroelastic fields for Hertz contact

After getting the contact parameters in Section 4, we now further solve for the magnetoelectroelastic fields for Hertz contact. For that it is useful if the elliptical coordinate system (ξ, ζ, η) is used. These elliptic coordinates are determined as the roots of the polynomial equation in υ given by

$$\frac{x^2}{a^2\upsilon^2} + \frac{y^2}{a^2(\upsilon^2 - e^2)} + \frac{z^2}{a^2(\upsilon^2 - 1)} = 1, \tag{44}$$

where $0 \leqslant \eta \leqslant e^2 \leqslant \zeta \leqslant 1 \leqslant \xi < \infty$.

5.1. Solutions for smooth contact

The contact stress and electric displacement inside the contact region are assumed as

$$p(r, \phi) = \frac{3P_z}{2\pi ab} \sqrt{1 - \frac{r^2 \cos^2 \phi}{a^2} - \frac{r^2 \sin^2 \phi}{b^2}},$$
$$d(r, \phi) = \frac{3Q}{2\pi ab} \sqrt{1 - \frac{r^2 \cos^2 \phi}{a^2} - \frac{r^2 \sin^2 \phi}{b^2}}, \quad 0 \leqslant r \leqslant C(\phi), \ \ 0 \leqslant \phi < 2\pi, \tag{45}$$

where a and b are determined by Eq. (38), $C(\phi)$ is the border of the contact region. For the elliptical contact region, $C(\phi)$ is in the following form

$$C(\phi) = ab \Big/ \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}. \tag{46}$$

Substituting $P_z = p(r_0, \phi_0) r_0 \, \mathrm{d}r_0 \, \mathrm{d}\phi_0$ and $Q = d(r_0, \phi_0) r_0 \, \mathrm{d}r_0 \, \mathrm{d}\phi_0$ into Eq. (31) and integrating the result over $0 \leqslant r_0 \leqslant C(\phi_0)$, $0 \leqslant \phi_0 \leqslant 2\pi$, the displacement functions become

$$\begin{aligned}
&\psi_0(r,\phi,z) = 0,\\
&\psi_j(r,\phi,z) = \frac{3(P_z\delta_j + Q\lambda_j)}{2\pi ab}\Re(r,\phi,z_j) \quad (j=1,2,3,4),
\end{aligned} \tag{47}$$

where

$$\Re(r,\phi,z_j) = \int_0^{2\pi}\int_0^{C(\phi_0)}\sqrt{1-\frac{r_0^2\cos^2\phi_0}{a^2}-\frac{r_0^2\sin^2\phi_0}{b^2}}\ln R_j^* r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \tag{48}$$

where R_j and R_j^* are defined in Eq. (32).

Substituting Eq. (47) into Eq. (13) and using the partial derivatives of $\Re(r,\phi,z_j)$ given by Hanson and Puja (1997), magnetoelectroelastic fields can be obtained as follows:

$$\begin{aligned}
U &= -\frac{3}{a^3}\sum_{j=1}^{4}(P_z\delta_j+Q\lambda_j)\{x[z_j\vartheta_1(\xi_j)-aI_{11}]+\mathrm{i}y[z_j\vartheta_2(\xi_j)-aI_{12}]\},\\
w_m &= \frac{3}{2a^3}\sum_{j=1}^{4}(P_z\delta_j+Q\lambda_j)s_jk_{mj}\{a^2F(\varphi_j,e)-x^2\vartheta_1(\xi_j)-y^2\vartheta_2(\xi_j)-z_j^2\vartheta_3(\xi_j)\},\\
\sigma_1 &= -\frac{6}{a^3}\sum_{j=1}^{4}(P_z\delta_j+Q\lambda_j)(c_{66}-\omega_{1j}s_j^2)z_j\vartheta_3(\xi_j),\\
\sigma_2 &= -\frac{6c_{66}}{a^4}\sum_{j=1}^{4}(P_z\delta_j+Q\lambda_j)\{az_j[\vartheta_1(\xi_j)-\vartheta_2(\xi_j)]+x^2I_8-y^2I_3+a^2(I_{12}-I_{11})+\mathrm{i}2xyI_4\},\\
\sigma_{zm} &= -\frac{3}{a^3}\sum_{j=1}^{4}(P_z\delta_j+Q\lambda_j)\omega_{mj}z_j\vartheta_3(\xi_j),\\
\tau_{zm} &= -\frac{3}{a^3}\sum_{j=1}^{4}(P_z\delta_j+Q\lambda_j)s_j\omega_{mj}[x\vartheta_1(\xi_j)+\mathrm{i}y\vartheta_2(\xi_j)],
\end{aligned} \tag{49}$$

where ξ_j $(j=1,2,3,4)$ are the complex elliptical coordinates which can be obtained by replacing z with z_j in Eq. (44) and satisfy $1 \leqslant \mathrm{Re}(\xi_j^2) < \infty$; $F(\varphi_j,e)(j=1,2,3,4)$ are the incomplete elliptic integrals of the first kind; φ_j $(j=1,2,3,4)$ and I_n $(n=3,4,8,11,12)$ are listed in Appendix A of Hanson and Puja (1997); $\vartheta_k(\xi_j)$ $(k=1,2,3;\ j=1,2,3,4)$ are same with $\psi_k(\xi_j)(k=1,2,3)$ in Appendix A of Hanson and Puja (1997).

5.2. Solutions for friction contact

When bodies ① and ② are also subjected to tangential loading causing them to slide on the surfaces of each other, it is assumed that the sliding friction could be determined by Coulomb friction law. Similarly, substituting following equation into Eq. (33)

$$P = \frac{3P_zf}{2\pi ab}\sqrt{1-\frac{r_0^2\cos^2\phi_0}{a^2}-\frac{r_0^2\sin^2\phi_0}{b^2}}r_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \quad f=f_x+\mathrm{i}f_y \tag{50}$$

and integrating the result over $0 \leqslant r_0 \leqslant C(\phi_0)$, $0 \leqslant \phi_0 \leqslant 2\pi$, the displacement functions become

$$\begin{aligned}
&\psi_0(r,\phi,z) = \mathrm{i}\frac{3P_zG_0}{2\pi ab}(f\bar{\Delta}-\bar{f}\Delta)[z_0\Re(r,\phi,z_0)-\Im(r,\phi,z_0)]\\
&\psi_j(r,\phi,z) = \frac{3P_zG_j}{2\pi ab}(f\bar{\Delta}+\bar{f}\Delta)[z_j\Re(r,\phi,z_j)-\Im(r,\phi,z_j)],
\end{aligned} \tag{51}$$

where $\Re(r,\phi,z_j)$ is defined in Eq. (48), and $\Im(r,\phi,z_j)$ is

$$\Im(r,\phi,z_j)=\int_0^{2\pi}\int_0^{C(\phi_0)}\sqrt{1-\frac{r_0^2\cos^2\phi_0}{a^2}-\frac{r_0^2\sin^2\phi_0}{b^2}}R_jr_0\,\mathrm{d}r_0\,\mathrm{d}\phi_0, \tag{52}$$

where R_j are expressed in Eq. (32). Substituting Eq. (51) into Eq. (13) and with using of the partial derivatives of $\Re(r,\phi,z_j)$ and $\Im(r,\phi,z_j)$ given by Hanson and Puja (1997), we obtain the magnetoelectroelastic field as follows:

$$\begin{aligned}
U={}&\frac{3P_z}{2a^4}\sum_{j=1}^{4}G_j\{-fa[a^2F(\varphi_j,e)-x^2\vartheta_1(\xi_j)-y^2\vartheta_2(\xi_j)-z_j^2\vartheta_3(\xi_j)]\\
&+\bar f[a(a^2-z_j^2)\vartheta_1(\xi_j)+a(z_j^2-b^2)\vartheta_2(\xi_j)-3ay^2I_1+a(y^2-x^2)I_2\\
&+2y^2z_jI_3-2x^2z_jI_8+3ax^2I_9+2a^2z_j(I_{11}-I_{12})+\mathrm{i}4xy(aI_2-z_jI_4)]\}\\
&-\frac{3P_z}{2a^4}G_0\{-fa[a^2F(\varphi_0,e)-x^2\vartheta_1(\xi_0)-y^2\vartheta_2(\xi_0)-z_0^2\vartheta_3(\xi_0)]\\
&-\bar f[a(a^2-z_0^2)\vartheta_1(\xi_0)+a[z_0^2-a^2(1-e^2)]\vartheta_2(\xi_0)-3ay^2I_1+a(y^2-x^2)I_2\\
&+2y^2z_0I_3-2x^2z_0I_8+3ax^2I_9+2a^2z_0(I_{11}-I_{12})+\mathrm{i}4xy(aI_2-z_0I_4)]\},\\
w_m={}&-\frac{6P_z}{a^3}\sum_{j=1}^{4}G_js_jk_{mj}\{xf_x[z_j\vartheta_1(\xi_j)-aI_{11}]+yf_y[z_j\vartheta_2(\xi_j)-aI_{12}]\},\\
\sigma_1={}&\frac{12P_z}{a^3}\sum_{j=1}^{4}G_j(c_{66}-\omega_{1j}s_j^2)[xf_x\vartheta_1(\xi_j)+yf_y\vartheta_2(\xi_j)],\\
\sigma_2={}&\frac{6c_{66}P_z}{a^4}\sum_{j=1}^{4}G_j\Big\{fa[x\vartheta_1(\xi_j)+\mathrm{i}y\vartheta_2(\xi_j)]+\bar f\Big[\frac{x}{a}\{3a^2(I_9-I_2)+3az_j(I_4-I_8)-x^2I_{10}+3y^2I_6\}\\
&+\mathrm{i}\frac{y}{a}\{3a^2(I_2-I_1)+3az_j(I_3-I_4)-3x^2I_7+y^2I_5\}\Big]\Big\}-\frac{6c_{66}P_z}{a^4}G_0\Big\{fa[x\vartheta_1(\xi_0)+\mathrm{i}y\vartheta_2(\xi_0)]\\
&-\bar f\Big[\frac{x}{a}\{3a^2(I_9-I_2)+3az_0(I_4-I_8)-x^2I_{10}+3y^2I_6\}+\mathrm{i}\frac{y}{a}\{3a^2(I_2-I_1)+3az_0(I_3-I_4)\\
&-3x^2I_7+y^2I_5\}\Big]\Big\},\\
\sigma_{zm}={}&\frac{6P_z}{a^3}\sum_{j=1}^{4}G_j\omega_{mj}[xf_x\vartheta_1(\xi_j)+yf_y\vartheta_2(\xi_j)],\\
\tau_{zm}={}&\frac{3P_z}{a^4}\sum_{j=1}^{4}G_js_j\omega_{mj}\{faz_j\vartheta_3(\xi_j)-\bar f[az_j\{\vartheta_1(\xi_j)-\vartheta_2(\xi_j)\}-a^2(I_{11}-I_{12})+x^2I_8-y^2I_3+\mathrm{i}2xyI_4]\}\\
&-\frac{3P_z}{a^4}G_0s_0\rho_m\{faz_0\vartheta_3(\xi_0)+\bar f[az_0\{\vartheta_1(\xi_0)-\vartheta_2(\xi_0)\}-a^2(I_{11}-I_{12})+x^2I_8-y^2I_3+\mathrm{i}2xyI_4]\},
\end{aligned} \tag{53}$$

where $F(\varphi_j, e)$ and ξ_j $(j = 0, 1, 2, 3, 4)$ are same as what they are in Eq. (49); φ_j $(j = 1, 2, 3, 4)$ and I_n $(n = 1, 2, 3, \ldots, 12)$ are listed in Appendices A and B in Hanson and Puja (1997). And $\vartheta_k(\xi_j)$ $(k = 1, 2, 3; j = 1, 2, 3, 4)$ are same as those of Eq. (49).

5.3. Numerical results for smooth elliptical contact

Assume $e = 3/5$ and suppose only force P_z acts on a body contacting with purely elastic, piezoelectric and magnetoelectroelastic half-spaces, respectively, which are assumed to be with the same elastic, piezoelectric and dielectric constants. The elastic and electric fields in the purely elastic, piezoelectric and magnetoelectroelastic half-spaces are compared to each other in Figs. 2–5 based on Hanson and Puja (1997); Ding et al. (2000) and Eq. (49), respectively. In addition, the magnetic components in the magnetoelectroelastic half-space are also shown in the figures. The material constants of magnetoelectroelastic half-space are shown in Table 1.

Symbols in figures are defined as follows:

$$\begin{aligned}
&{}^n\sigma_x = \frac{\sigma_x}{p_m}, \quad {}^n\sigma_y = \frac{\sigma_y}{p_m}, \quad {}^n\sigma_z = \frac{\sigma_z}{p_m}, \quad {}^n\tau_1 = \frac{\tau_1}{p_m}, \quad p_m = \frac{P_z}{\pi ab}, \\
&{}^n\Phi = \frac{\Phi}{\Phi_m}, \quad {}^nD_x = \frac{D_x}{D_m}, \quad {}^nD_y = \frac{D_y}{D_m}, \quad {}^nD_z = \frac{D_z}{D_m}, \quad \Phi_m = \frac{P_z}{a \times 10^2}, \quad D_m = \frac{P_z}{a^2 \times 10^{10}}, \\
&{}^n\Psi = \frac{\Psi}{\Psi_m}, \quad {}^nB_x = \frac{B_x}{B_m}, \quad {}^nB_y = \frac{B_y}{B_m}, \quad {}^nB_z = \frac{B_z}{B_m}, \quad \Psi_m = \frac{P_z}{a \times 10^5}, \quad B_m = \frac{P_z}{a^2 \times 10^{10}},
\end{aligned} \tag{54}$$

where $\tau_1 = (\sigma_{\max} - \sigma_{\min})/2$ is the maximum shear stress at a point.

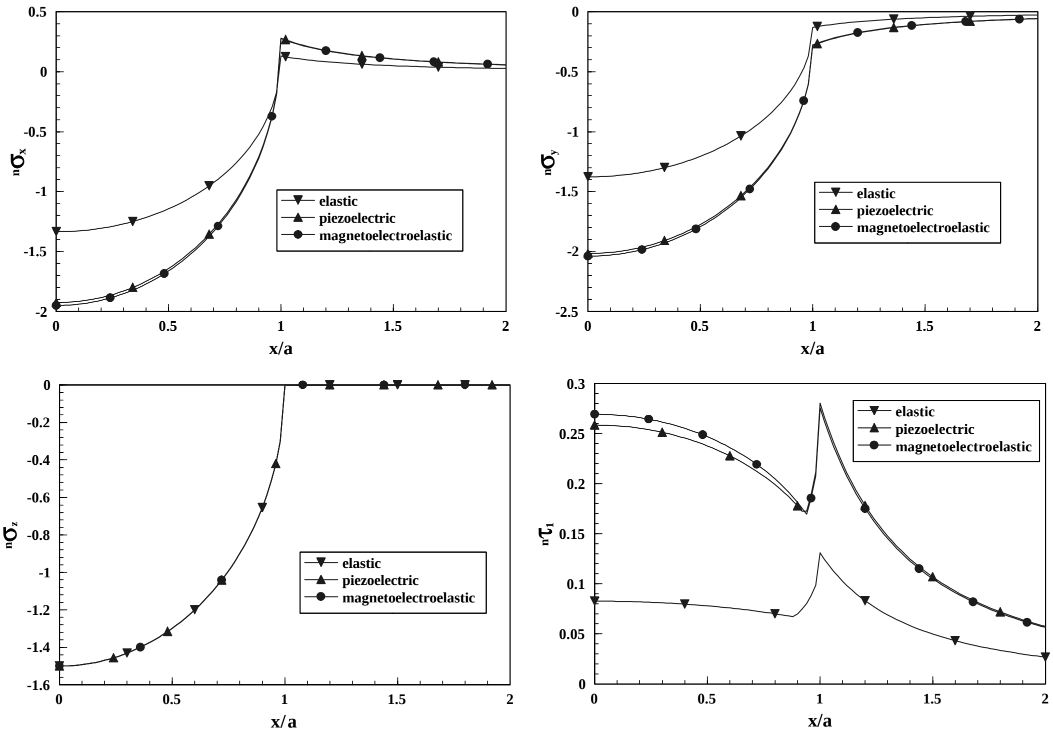

Fig. 2. Elastic field on the half-major axis of contact ellipse.

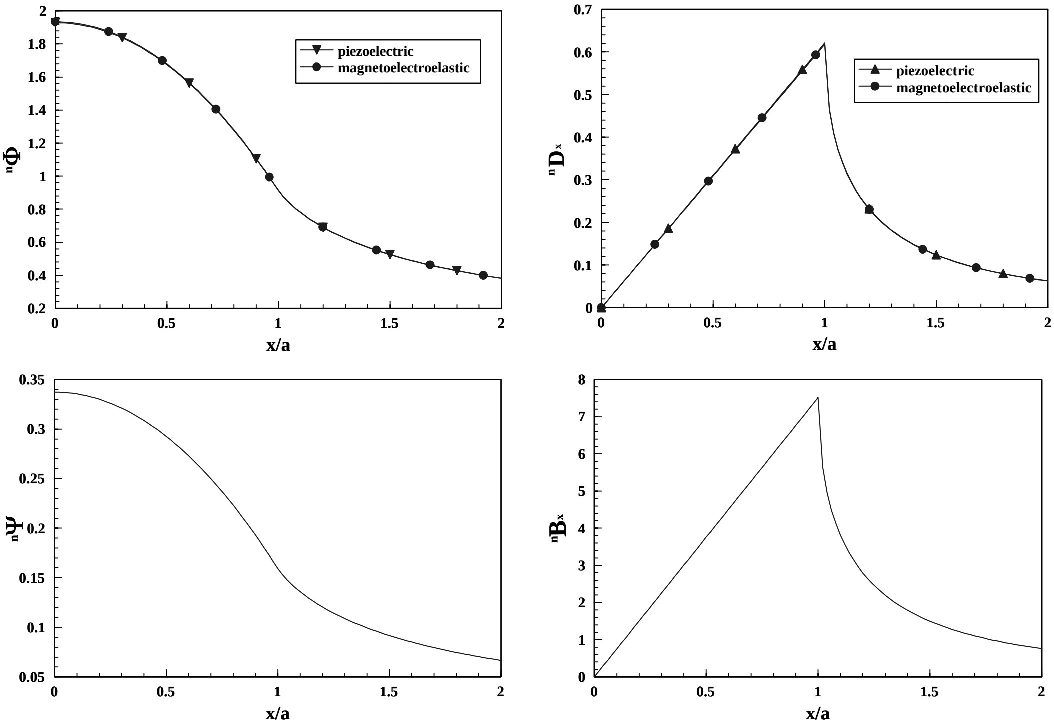

Fig. 3. Electromagnetic field on the half-major axis of contact ellipse.

From above-mentioned figures, we can see that the elastic and electric fields in the magneto-electroelastic half-space are similar with those of corresponding piezoelectric and purely elastic half-space.

1. For the three kinds of half-space, the points on the major axis of the elliptical contact region are nearly all in the state of being pressed at three orthogonal directions. The greatest value of normal stresses for magnetoelectroelastic half-space is larger than that of corresponding piezoelectric and purely elastic half-spaces, and all occur at the center of contact ellipse.
2. The greatest maximum shear stress in magnetoelectroelastic half-space is $^n\tau_1 = 0.4989p_m$ and occurs in the symmetric axis at the depth of $z = 0.46a$. The greatest maximum shear stress in piezoelectric half-space is $^n\tau_1 = 0.5006p_m$ and occurs in the symmetric axis at the depth of $z = 0.44a$. The greatest maximum shear stress in purely elastic half-space is $^n\tau_1 = 0.4527p_m$ and also occurs in the symmetric axis at the depth of $z = 0.44a$.
3. The greatest electric displacement D_z in magnetoelectroelastic half-space is $^nD_z = -0.2159D_m$ and occurs in the symmetric axis at the depth of $z = 0.36a$. The greatest electric displacement D_z in piezoelectric half-space is $^nD_z = -0.2177D_m$ and occurs in the symmetric axis at the depth of $z = 0.36a$. The D_x reach the peak values of $^nD_x = 0.6178D_m$ and $^nD_x = 0.6211D_m$ at the point of $(a, 0, 0)$ of magnetoelectroelastic and piezoelectric half-spaces, respectively.
4. The greatest magnetic induction ^nB_z in magnetoelectroelastic half-space is $^nB_z = 1.9566B_m$ and occurs in the symmetric axis at the depth of $z = 0.02a$. The B_x reach the peak values of $^nB_x = 7.5141B_m$ at the point of $(a, 0, 0)$ of magnetoelectroelastic half-spaces.

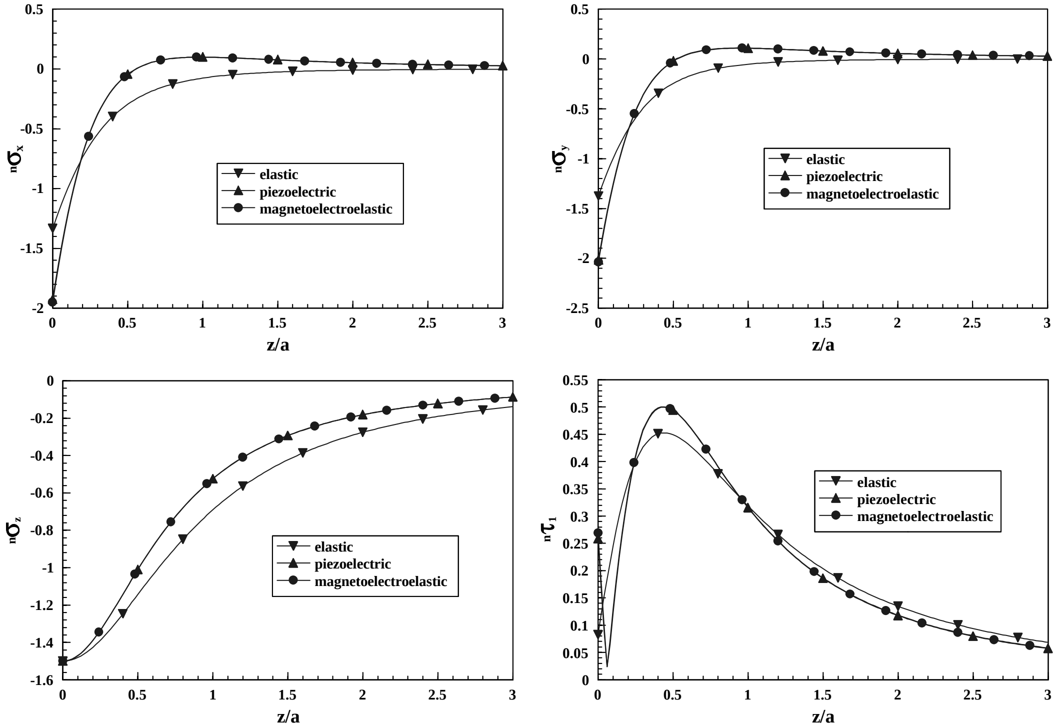

Fig. 4. Elastic field on the symmetric axis.

In additions, calculations also show us that there are ${}^{n}\sigma_z = {}^{n}\tau_{zx} = {}^{n}\tau_{zy} = 0$ on the border of elliptical contact region and the maximum and minimum principle stresses in the xoy plane are contrary sign and equal in values, so every point on this border is actually in the state of pure shear for three kinds of half-spaces. The stress distributions on the minor axis of contact ellipse is similar to those on the major axis for three kinds of half-spaces and so does the electromagnetic field on the minor axis for magnetoelectroelastic and piezoelectric half-spaces. In addition, there are ${}^{n}D_y = {}^{n}D_z = 0$ on the major axis, ${}^{n}D_x = {}^{n}D_z = 0$ and ${}^{n}B_x = {}^{n}B_z = 0$ on the minor axis, ${}^{n}D_z = 0$ and ${}^{n}B_z = 0$ on the elliptical contact border and ${}^{n}D_x = {}^{n}D_y = 0$ and ${}^{n}B_x = {}^{n}B_y = 0$ on the symmetric axis for magnetoelectroelastic and piezoelectric half-spaces.

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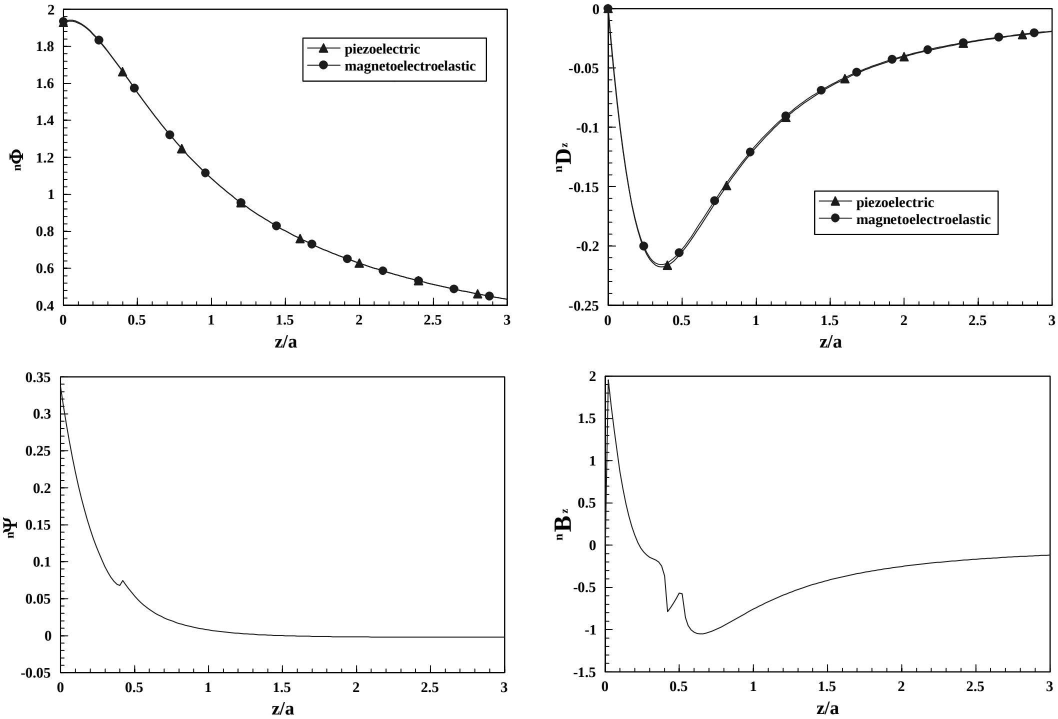

Fig. 5. Electromagnetic field on the symmetric axis.

Table 1
Physical constants of magnetoelectroelastic material (refer to Li (2000))

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	g_{11}
2.86×10^{11}	1.73×10^{11}	1.70×10^{11}	2.695×10^{11}	4.53×10^{10}	5.0×10^{-12}
e_{15}	e_{31}	e_{33}	ε_{11}	ε_{33}	g_{33}
11.6	−4.4	18.6	8.0×10^{-11}	9.3×10^{-11}	3.0×10^{-12}
d_{15}	d_{31}	d_{33}	μ_{11}	μ_{33}	
550	580.3	699.7	-5.90×10^{-4}	1.57×10^{-4}	

Units: elastic constants, $\mathrm{N\,m^{-2}}$; piezoelectric constants, $\mathrm{C\,m^{-2}}$; piezomagnetic constants, $\mathrm{N\,A^{-1}\,m^{-1}}$; dielectric constants, $\mathrm{C^2\,N^{-1}\,m^{-2}}$; electromagnetic constants, $\mathrm{N\,s\,V^{-1}\,C^{-1}}$; magnetic constants, $\mathrm{N\,s^2\,C^{-2}}$.

Appendix A

$a_k (k = 1, 2, 3, 4, 5)$ in Eq. (11) are defined as follows:

$$
\begin{aligned}
a_1 &= c_{44}[c_{33}(\varepsilon_{33}\mu_{33} - g_{33}^2) - 2e_{33}g_{33}d_{33} + \mu_{33}e_{33}^2 + \varepsilon_{33}d_{33}^2],\\
a_2 &= c_{11}[c_{33}(\varepsilon_{33}\mu_{33} - g_{33}^2) - 2e_{33}g_{33}d_{33} + \mu_{33}e_{33}^2 + \varepsilon_{33}d_{33}^2] + c_{44}[c_{44}(\varepsilon_{33}\mu_{33} - g_{33}^2)\\
&\quad + c_{33}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) - 2e_{15}g_{33}d_{33} - 2e_{33}(g_{11}d_{33} + g_{33}d_{15})\\
&\quad + (\mu_{11}e_{33}^2 + 2\mu_{33}e_{15}e_{33}) + (\varepsilon_{11}d_{33}^2 + 2\varepsilon_{33}d_{15}d_{33})]\\
&\quad - (c_{13} + c_{44})[(c_{13} + c_{44})(\varepsilon_{33}\mu_{33} - g_{33}^2) + (e_{15} + e_{31})(e_{33}\mu_{33} - d_{33}g_{33})\\
&\quad - (d_{15} + d_{31})(e_{33}g_{33} - d_{33}\varepsilon_{33})] - (e_{15} + e_{31})[(c_{13} + c_{44})(e_{33}\mu_{33} - g_{33}d_{33})\\
&\quad - (e_{15} + e_{31})(c_{33}\mu_{33} + d_{33}^2) + (d_{15} + d_{31})(c_{33}g_{33} + d_{33}e_{33})]\\
&\quad - (d_{15} + d_{31})[(c_{13} + c_{44})(-e_{33}g_{33} + \varepsilon_{33}d_{33}) + (e_{15} + e_{31})(c_{33}g_{33} + e_{33}d_{33})\\
&\quad - (d_{15} + d_{31})(c_{33}\varepsilon_{33} + e_{33}^2)],\\
a_3 &= c_{11}[c_{44}(\varepsilon_{33}\mu_{33} - g_{33}^2) + c_{33}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) - 2e_{15}g_{33}d_{33}\\
&\quad - 2e_{33}(g_{11}d_{33} + g_{33}d_{15}) + (\mu_{11}e_{33}^2 + 2\mu_{33}e_{15}e_{33}) + (\varepsilon_{11}d_{33}^2 + 2\varepsilon_{33}d_{15}d_{33})]\\
&\quad + c_{44}[c_{44}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + c_{33}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}(g_{11}d_{33} + g_{33}d_{15})\\
&\quad - 2e_{33}g_{11}d_{15} + 2\mu_{11}e_{15}e_{33} + \mu_{33}e_{15}^2 + 2\varepsilon_{11}d_{15}d_{33} + \varepsilon_{33}d_{15}^2]\\
&\quad - (c_{13} + c_{44})[(c_{13} + c_{44})(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + (e_{15} + e_{31})\\
&\quad \times (e_{15}\mu_{33} + e_{33}\mu_{11} - d_{15}g_{33} - d_{33}g_{11}) - (d_{15} + d_{31})(e_{15}g_{33} + e_{33}g_{11} - d_{15}\varepsilon_{33} - d_{33}\varepsilon_{11})]\\
&\quad - (e_{15} + e_{31})[(c_{13} + c_{44})(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15})\\
&\quad - (e_{15} + e_{31})(c_{44}\mu_{33} + c_{33}\mu_{11} + 2d_{15}d_{33}) + (d_{15} + d_{31})\\
&\quad \times (c_{44}g_{33} + c_{33}g_{11} + d_{15}e_{33} + d_{33}e_{15})]\\
&\quad - (d_{15} + d_{31})[(c_{13} + c_{44})(-e_{15}g_{33} - e_{33}g_{11} + \varepsilon_{11}d_{33} + \varepsilon_{33}d_{15})\\
&\quad + (e_{15} + e_{31})(c_{44}g_{33} + c_{33}g_{11} + e_{15}d_{33} + e_{33}d_{15})\\
&\quad - (d_{15} + d_{31})(c_{44}\varepsilon_{33} + c_{33}\varepsilon_{11} + 2e_{15}e_{33})],\\
a_4 &= c_{11}[c_{44}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + c_{33}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}(g_{11}d_{33} + g_{33}d_{15})\\
&\quad - 2e_{33}g_{11}d_{15} + 2\mu_{11}e_{15}e_{33} + \mu_{33}e_{15}^2 + 2\varepsilon_{11}d_{15}d_{33} + \varepsilon_{33}d_{15}^2]\\
&\quad + c_{44}[c_{44}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}g_{11}d_{15} + \mu_{11}e_{15}^2 + \varepsilon_{11}d_{15}^2]\\
&\quad - (c_{13} + c_{44})[(c_{13} + c_{44})(\varepsilon_{11}\mu_{11} - g_{11}^2) + (e_{15} + e_{31})(e_{15}\mu_{11} - d_{15}g_{11})\\
&\quad - (d_{15} + d_{31})(e_{15}g_{11} - d_{15}\varepsilon_{11})] - (e_{15} + e_{31})[(c_{13} + c_{44})(e_{15}\mu_{11} - g_{11}d_{15})\\
&\quad - (e_{15} + e_{31})(c_{44}\mu_{11} + d_{15}^2) + (d_{15} + d_{31})(c_{44}g_{11} + d_{15}e_{15})]\\
&\quad - (d_{15} + d_{31})[(c_{13} + c_{44})(-e_{15}g_{11} + \varepsilon_{11}d_{15}) + (e_{15} + e_{31})(c_{44}g_{11} + e_{15}d_{15})\\
&\quad - (d_{15} + d_{31})(c_{44}\varepsilon_{11} + e_{15}^2)],\\
a_5 &= c_{11}[c_{44}(\varepsilon_{11}\mu_{11} - g_{11}^2) - 2e_{15}g_{11}d_{15} + \mu_{11}e_{15}^2 + \varepsilon_{11}d_{15}^2].
\end{aligned}
\tag{A.1}
$$

Appendix B

n_1, n_2, n_3, m_{4m}, m_{5m}, m_{6m} and m_{7m} $(m = 1, 2, 3)$ in Eq. (15) are defined as follows:

$$\begin{aligned}
n_1 &= (c_{13}+c_{44})(\varepsilon_{11}\mu_{11}-g_{11}^2)+(e_{15}+e_{31})(e_{15}\mu_{11}-g_{11}d_{15})-(d_{15}+d_{31})(e_{15}g_{11}-\varepsilon_{11}d_{15}),\\
n_2 &= (c_{13}+c_{44})(\varepsilon_{11}\mu_{33}+\varepsilon_{33}\mu_{11}-2g_{11}g_{33})+(e_{15}+e_{31})(e_{15}\mu_{33}+e_{33}\mu_{11}-g_{11}d_{33}-g_{33}d_{15})\\
&\quad -(d_{15}+d_{31})(e_{15}g_{33}+e_{33}g_{11}-\varepsilon_{11}d_{33}-\varepsilon_{33}d_{15}),\\
n_3 &= (c_{13}+c_{44})(\varepsilon_{33}\mu_{33}-g_{33}^2)+(e_{15}+e_{31})(e_{33}\mu_{33}-g_{33}d_{33})-(d_{15}+d_{31})(e_{33}g_{33}-\varepsilon_{33}d_{33}),\\
n_{41} &= c_{11}(\varepsilon_{11}\mu_{11}-g_{11}^2),\\
n_{51} &= c_{11}(\varepsilon_{11}\mu_{33}+\varepsilon_{33}\mu_{11}-2g_{11}g_{33})+c_{44}(\varepsilon_{11}\mu_{11}-g_{11}^2)+\mu_{11}(e_{15}+e_{31})^2+\varepsilon_{11}(d_{15}+d_{31})^2\\
&\quad -2g_{11}(e_{15}+e_{31})(d_{15}+d_{31}),\\
n_{61} &= c_{11}(\varepsilon_{33}\mu_{33}-g_{33}^2)+c_{44}(\varepsilon_{11}\mu_{33}+\varepsilon_{33}\mu_{11}-2g_{11}g_{33})+\mu_{33}(e_{15}+e_{31})^2+\varepsilon_{33}(d_{15}+d_{31})^2\\
&\quad -2g_{33}(e_{15}+e_{31})(d_{15}+d_{31}),\\
n_{71} &= c_{44}(\varepsilon_{33}\mu_{33}-g_{33}^2),\\
n_{42} &= c_{11}(e_{15}\mu_{11}-g_{11}d_{15}),\\
n_{52} &= c_{11}(e_{15}\mu_{33}+e_{33}\mu_{11}-g_{11}d_{33}-g_{33}d_{15})+c_{44}(e_{15}\mu_{11}-g_{11}d_{15})\\
&\quad -(e_{15}+e_{31})[\mu_{11}(c_{13}+c_{44})+d_{15}(d_{15}+d_{31})]+(d_{15}+d_{31})[g_{11}(c_{13}+c_{44})+e_{15}(d_{15}+d_{31})],\\
n_{62} &= c_{11}(e_{33}\mu_{33}-g_{33}d_{33})+c_{44}(e_{15}\mu_{33}+e_{33}\mu_{11}-g_{11}d_{33}-g_{33}d_{15})\\
&\quad -(e_{15}+e_{31})[\mu_{33}(c_{13}+c_{44})+d_{33}(d_{15}+d_{31})]+(d_{15}+d_{31})[g_{33}(c_{13}+c_{44})+e_{33}(d_{15}+d_{31})],\\
n_{72} &= c_{44}(e_{33}\mu_{33}-g_{33}d_{33}),\\
n_{43} &= c_{11}(-e_{15}g_{11}+\varepsilon_{11}d_{15}),\\
n_{53} &= c_{11}(-e_{15}g_{33}-e_{33}g_{11}+\varepsilon_{11}d_{33}+\varepsilon_{33}d_{15})+c_{44}(-e_{15}g_{11}+\varepsilon_{11}d_{15})\\
&\quad +(e_{15}+e_{31})[g_{11}(c_{13}+c_{44})+d_{15}(e_{15}+e_{31})]-(d_{15}+d_{31})[\varepsilon_{11}(c_{13}+c_{44})+e_{15}(e_{15}+e_{31})],\\
n_{63} &= c_{11}(-e_{33}g_{33}+\varepsilon_{33}d_{33})+c_{44}(-e_{15}g_{33}-e_{33}g_{11}+\varepsilon_{11}d_{33}+\varepsilon_{33}d_{15})\\
&\quad +(e_{15}+e_{31})[g_{33}(c_{13}+c_{44})+d_{33}(e_{15}+e_{31})]-(d_{15}+d_{31})[\varepsilon_{33}(c_{13}+c_{44})+e_{33}(e_{15}+e_{31})],\\
n_{73} &= c_{44}(-e_{33}g_{33}+\varepsilon_{33}d_{33}).
\end{aligned} \tag{B.1}$$

References

Alshits, V.I., Darinskii, A.N., Lothe, J., 1992. On the existence of surface waves in half-anisotropic elastic media with piezoelectric and piezomagnetic properties. Wave Motion 16, 265–283.

Chen, W.Q., Ding, H.J., 1999. Indentation of a transversely isotropic piezoelectric half-space by a rigid sphere. Acta Mech. Solid. Sin. 12, 114–120.

Chen, W.Q., Shioya, T., Ding, H.J., 1999. The elasto-electric field for a rigid conical punch on a transversely isotropic piezoelectric half-space. J. Appl. Mech. 66, 764–771.

Chen, W.Q., 1999. Inclined circular flat punch on a transversely isotropic piezoelectric half-space. Arch. Appl. Mech. 69, 455–464.

Chen, W.Q., 2000. On piezoelastic contact problem for a smooth punch. Int. J. Solids Struct. 37, 2331–2340.

Chen, W.T., 1969. Stresses in some anisotropic materials due to indentation and sliding. Int. J. Solids Struct. 5, 191–214.

Conway, H.D., Farnham, K.A., 1967. Shear stresses induced on an axis of symmetry by circular sliding contact on a transversely isotropic body. J. Appl. Mech. 34, 756–757.

Conway, H.D., Farnham, K.A., Ku, T.C., 1967. The indentation of a transversely isotropic half-space by a rigid sphere. J. Appl. Mech. 34, 491–492.

Ding, H.J., Chen, B., Liang, J., 1996. General solutions for coupled equations for piezoelectric media. Int. J. Solids Struct. 33, 2283–2298.

Ding, H.J. et al., 1997. The Elastic Mechanics of Transverse isotropy. Zhejiang University Press, PR China (in Chinese).

Ding, H.J., Hou, P.F., Guo, F.L., 1999. Elastic and electric fields for elliptical contact for transversely isotropic piezoelectric bodies. J. Appl. Mech. 66, 560–562.

Ding, H.J., Hou, P.F., Guo, F.L., 2000. The elastic and electric fields for three-dimensional contact for transversely isotropic piezoelectric materials. Int. J. Solids Struct. 37, 3201–3229.

Elliott, H.A., 1948. Three-dimensional stress distributions in hexagonal aeolotropic crystals. Proc. Camb. Philos. Soc. 44, 522–553.

Elliott, H.A., 1949. Axialsymmetric stress distributions in aeolotropic hexagonal crystals: the problem of the plane and related problem. Proc. Camb. Philos. Soc. 45, 621–630.

Fabrikant, V.I., 1989. Applications of Potential Theory in Mechanics. Selection of New Results. Kluwer Academic Publishers, The Netherlands.

Fabrikant, V.I., 1991. Mixed Boundary value Problems of Potential Theory and Their Applications in Engineering. Kluwer Academic Publishers, The Netherlands.

Fan, H., Sze, K.Y., Yang, W., 1996. Two-dimensional contact on a piezoelectric half-space. Int. J. Solids Struct. 33, 1305–1315.

Giannakopoulos, A.E., 2000. Strength analysis of spherical indentation of piezoelectric materials. J. Appl. Mech. 67, 409–416.

Gladwell, G.M.L., 1980. Contact Problems in the Classical Theory of Elasticity. Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands.

Green, A.E., Zerna, W., 1954. Theoretical Elasticity. Clarendon Press, Oxford.

Hanson, M.T., 1992a. The elastic field for spherical Hertzian contact including sliding friction for transversely isotropy. J. Tribol. 114, 606–611.

Hanson, M.T., 1992b. The elastic field for conical indentation including sliding friction for transverse isotropy. J. Appl. Mech. 59, S123–S130.

Hanson, M.T., 1994. The elastic field for an upright or tilted sliding circular flat punch on a transversely isotropic half-space. Int. J. Solids Struct. 31, 567–586.

Hanson, M.T., Puja, I.W., 1997. The elastic field resulting from elliptical Hertzian contact of transversely isotropic bodies: closed-form solutions for normal and shear loading. J. Appl. Mech. 64, 457–465.

Hertz, H., 1882. Vber Die Beruhrung Fester Elastischer Korper (On the contact of elastic solids). J. Reine Angew. Math. 92, 156–171.

Hu, H.C., 1953. On three-dimensional problems of the theory of elasticity of a transversely isotropic body. Acta. Scient. Sin. 2 (2), 145–151.

Huang, J.H., Kuo, W.S., 1997. The analysis of piezoelectric/piezomagnetic composite materials containing an ellipsoidal inclusion. J. Appl. Phys. 83, 1378–1386.

Johnson, K.L., 1985. Contact Mechanics. Cambridge University Press, Cambridge.

Keer, L.M., Mowry, D.B., 1979. The stress field created by a circular sliding contact on transversely isotropic spheres. Int. J. Solids Struct. 15, 33–39.

Li, J.Y., Dunn, M.L., 1998. Micromechanics of magnetoelectroelastic composite materials: average fields and effective behavior. J. Intell. Mater. Syst. Struct. 7, 404–416.

Li, J.Y., 2000. Magnetoelectroelastic multi-inclusion and inhomogeneity problems and their applications in composite materials. Int. J. Eng. Sci. 38, 1993–2011.

Lin, W., Kuo, C.H., Keer, L.M., 1991. Analysis of a transversely isotropic half-space under normal and tangential loadings. J. Tribol. 113, 335–338.

Mindlin, R.D., 1949. Compliance of elastic bodies in contact. J. Appl. Mech. 16, 259–268.

Muskhelishvili, N.I., 1953. Some Basic Problems of the Mathematical Theory of Elasticity. Groningen, Noordhoff.

Pan, Y.C., Chou, T.W., 1976. Point Force Solution for an Infinite Transversely Isotriopic Solids. J. Appl. Mech. 43, 608–612.

Shield, R.T., 1951. Notes on problems in hexagonal aeolotropic materials. Proc. Camb. Philos. Soc. 47, 401–409.

Sneddon, I.N., 1951. Fourier Transforms. McGraw-Hill Book Company, New York.

Sridhar, S., Giannakopoulos, A.E., Suresh, S., 2000. Mechanical and electrical responses of piezoelectric solids to conical indentation. J. Appl. Phys. 87, 8451–8456.

Willis, J.R., 1966. Hertzian contact of anisotropic bodies. J. Mech. Phys. Solids 14, 163–176.

Willis, J.R., 1967. Boussinesq problem for an anisotropic half-space. J. Mech. Phys. Solids 15, 331–339.